

Arbitrary Rate Permutation Modulation for the Gaussian Channel

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Abstract—In this paper non-group permutation modulated sequences for the Gaussian channel are considered. Without the restriction to group codes rather than subsets of group codes, arbitrary rates are achievable. The code construction utilizes the known optimal group constellations to ensure at least the same performance but exploit the Gray code ordering structure of multiset permutations as a selection criterion at the decoder. The decoder achieves near maximum likelihood performance at low computational cost and low additional memory requirements at the receiver.

I. INTRODUCTION

The history of permutation modulation goes back more than forty years where it was first introduced by Slepian in [1] and more generally in the framework of group codes in [2]. The appropriate definitions are

Definition I.1

Let x denote an unit vector in \mathbb{R}^n and G a finite group (order $M = |G|$) of orthogonal n -by- n matrices (or a finite orthogonal representation of an abstract group).

- 1) The pair (G, x) determines the (M, n) group code with initial vector x defined as $Gx = \{gx \mid g \in G\}$
- 2) If G is a permutation group of degree n then the corresponding group code is called *permutation code*
- 3) If G is the full symmetric group S_n of degree n then the corresponding permutation code is called *permutation modulation*

Since G consists of orthogonal matrices, $\|gx\| = \|x\| = 1$ for all $g \in G$, each codeword can be identified with a point on a $(n - 1)$ dimensional sphere. Thus permutation modulation generates spherical (or equal energy) codes. Usually there are two variants of permutation modulation in the literature, but here we are concerned with variant I only as defined above^a. It is explicitly specified as follows [1]:

$$x := (\mu_1^{(m_1)}, \dots, \mu_k^{(m_k)}) \quad (1)$$

where $\mu_1 < \dots < \mu_k$, $\mu^{(i)}$ denotes i repetitions of the value μ , and $n = m_1 + \dots + m_k$. The code is given by the set of all

^a)Variant II defined in [1] can be obtained from variant I by applying all possible sign changes in the components of each gx , $g \in G$, whereas now the components of the initial vector x are assumed to be non-negative

distinct permutations of the components of the initial vector x . Thus the size of the code is given as

$$M := \frac{n!}{m_1! \dots m_k!} = \frac{|m|!}{m!} \quad (2)$$

where the notation on the right hand side has been borrowed from the multi index notation applied to the vector $m = (m_1, \dots, m_k)$, whereas $|m| := m_1 + \dots + m_k$ and $m! := m_1! \dots m_k!$. In this context x can be interpreted as a multiset, i.e. the set of its components with repetitions and the corresponding permutations are called multiset permutations. The repetitions represented by the vector m have been introduced to bound the cardinality M of distinct permutations away from $n!$, and therefore to support a variety of code rates

$$R = \frac{1}{n} \log_2(M) \quad (3)$$

for a fixed sequence length n .

In this setting two prominent questions have been asked

- Q1) Given a group G , how to design the initial vector x , such that the resulting group code has maximum minimal distance? (Here it is assumed that each codeword has equal probability, such that the distance distribution characterizes the maximum likelihood detection)
- Q2) How to decode efficiently? (Note that the size M in (2) is still large in most cases, such that maximum likelihood decoding is not practical)

The first question is known as the *initial vector problem* (IVP) and has been addressed in [1] through numerical search only. The further history of the IVP covers in particular the following items: In [3] it has been shown that for so-called full homogeneous components of group representations the IVP can be solved. In the more explicit setting of $G = S_n$ and prescribed x of the form (1) with the vector m given in advance an optimal solution (i.e. determining the vector $\mu = (\mu_1, \dots, \mu_k)$) has been obtained in [4]. An algorithm for the general case (arbitrary group G) based on mathematical programming has been introduced in [5] and refined in [6] by means of generalized geometric programming^b. An explicit analytic solution of the IVP for permutation modulation ($G = S_n$) has been presented in a hardly recognized paper [8]: Given (n, k) the optimal initial vector (1) is determined such that

^b)This method is described in detail in [7], where the authors also remarked, that the algorithm does not guarantee to find the global optimum.

the signal energy is minimized (in particular the vector m is not required to be given in advance as in [4]). Due to the Lagrangian approach this method finds local minima, but parametrizes all possible solutions.

For the second question there has been developed an optimal solution in [1] for the case of permutation modulation: The receiver replaces the first m_1 smallest components of the received vector by μ_1 , the m_2 secondary smallest components by μ_2 , and so on. For later reference let us denote this procedure as the function SLEPIANDETECT, which takes a sequence of length n as input and outputs the sequence just constructed. For an arbitrary group G a decoding algorithm has been developed in [9], which refines the function SLEPIANDETECT by performing an iterative search afterwards. Moreover, in [10] ranking/unranking (resp. demapper/mapper) algorithms for multiset permutations with respect to lexicographical order are presented, which allow to convert information bit sequences to codewords and vice versa.

Contribution: In this paper the following achievements with respect to permutation modulation (variant I) corresponding to the questions Q1 and Q2 are presented:

- A1) The solutions [5], [8] to the initial vector problem provide only quantized rates corresponding to the cardinalities of subgroups of the permutation group S_n (or the choice of the initial vector respectively), i.e. M is exclusively given by (2). Since M is usually still very large, only high rates according to (3) can be achieved in this way. In this paper a permutation modulation construction scheme allowing *arbitrary rates* while maintaining large minimal distances is presented. In particular codes with not too large rates in possibly high dimensions n will be constructed. These new codes do not possess any group (orbit) structure any longer, but can be easily constructed.
- A2) A low complex suboptimal decoder for these codes is presented. This decoder, though obtained in a completely different manner, shares some properties of the algorithm in [9] and also with [11]. One part of the algorithm is the adaption of the mapping/demapping functions in [10].

As a further motivation for the particular code constructions presented here serves a certain application of space-time code design described in [12], where space-time block codes are transformed into spherical codes. In this situation a target (space-time code) rate has been specified together with a (possibly large) space-time code block length T . This code is then transformed into a spherical code with sequence length $n > T$, thus the spherical code rate is scaled by a factor of T/n and one ends up with a quite small rate in a high dimensional sphere which does not necessary fulfill the requirement (3).

II. RATE ADAPTED CODE CONSTRUCTION

The construction exploits the results obtained in [8] about the structure of the optimal initial vector (1)^c: For prescribed

^cNote that the solution is based on some Lagrangian optimization technique with discretized constraints. Although not mentioned in [8], the proof given exploits the convexity of the Lagrangian functional f to adapt the method to the discretized case

k and the minimal distance held fixed the optimal initial vector is a minimizer of the energy functional

$$E(m, \mu) = \sum_{i=1}^k m_i \mu_i^2 \quad (4)$$

The solution is [8]

$$\mu_i = -\frac{k-1}{2} + (i-1), \quad i = 1..k \quad (5)$$

$$m_i = \lfloor e^{-(\mu_i^2 + \eta)/\lambda} \rfloor$$

where $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer and $\lambda > 0$, $\eta < 0$ are some kind of Lagrangian parameters which parameterize the space of solutions unless (n, R) is fixed according to (3). The corresponding permutation modulation is $S_n \hat{x}$, where $\hat{x} := x/\|x\|$.

Let us now fix some target rate R and a corresponding code size $N = \lceil 2^{nR} \rceil$ ($\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ denote the common rounding functions to the next greater resp. smaller integer number). Then taking a solution (5) with rate greater than R and corresponding size M subject to (2) will be the starting configuration for the next step. This step utilizes ranking algorithms for multiset permutations. The idea is to find an ordering of multiset permutations which reflects the Euclidean distance relations between the corresponding code points. This is roughly achieved by some Gray code ordering. An algorithm which lists all M multiset permutations with respect to this ordering can be obtained online from [13]. Having this list at hand, select N out of the total of M elements of the list as equidistant as possible. To this end define $N_0 \in \{0, 1, \dots, N\}$ to be the maximum number which satisfies $\lceil M/N \rceil (N - N_0 - 1) + \lfloor M/N \rfloor N_0 \leq M - 1$ and set $n_0 := \lceil M/N \rceil (N - N_0 - 1)$. With this settings pick the first $N - N_0$ elements in the list equidistantly spaced by $\lceil M/N \rceil$ and the remaining N_0 ones starting at position n_0 equidistantly spaced by $\lfloor M/N \rfloor$. The result is an injective mapping

$$\text{MPGRAYENCODE} : \{0, \dots, N-1\} \longrightarrow \{0, \dots, M-1\} \quad (6)$$

which parameterizes N out of the M multiset permutations with largest possible Gray ordering separation. Let us denote the set of N selected multiset permutations by \mathcal{MP} and the corresponding (N, n) code by $\mathcal{C} \subset S_n \hat{x}$. Clearly the minimum distance of \mathcal{C} is at least as large as the minimum distance of $S_n \hat{x}$ and due to the correspondence between Gray ordering of permutations and Euclidean distances of codewords, the simple parameterization (6) seems promising in order to achieve a large minimal distance for \mathcal{C} .

To allow for a low complex decoding at the receiver we need a translation table, which translates lexicographic ranks to Gray code ranks. The need arises because although we have the algorithm [13] which lists all multiset permutations in Gray code order, there is no corresponding ranking function available (at least to the knowledge of the author). A rank function for (multiset) permutations is a function, which assigns to each (multiset) permutation a unique number in the range $0, \dots, n!(M)$ and establishes an ordering of

(multiset) permutations. The inverse mapping is called an unrank function. For ordinary permutations there exist rank and unrank functions with respect to different ordering criteria, including lexicographic and Gray code ordering. For multiset permutations lexicographic rank and unrank functions have been presented in [10, function Demapping/Mapping]^{d)} with average complexity proportional to $n^k/2$. Let us denote the lexicographic rank function by LEXRANK. It establishes the required translation table

$$\text{LEX2GRAY} : \{0, \dots, M-1\} \longrightarrow \{0, \dots, M-1\} \quad (7)$$

as the inverse mapping of $i \longmapsto \text{LEXRANK}(\pi(i))$ when $\pi(i)$ is the multiset permutation corresponding to $i \in \{0, \dots, M-1\}$ with respect to the given Gray ordered list and $\text{LEXRANK}(i)$ the lexicographic rank. This mapping can be stored efficiently as a list of M integer values at the receiver.

III. FAST NEAR ML DECODING

The transmission of data proceeds as follows. nR information bits are mapped to one of the N codewords. Let us assume message i is assigned to codeword $x(i) \in \mathcal{C}$ given by multiset permutation no. i in \mathcal{MP} applied to the initial vector x . $x(i)$ will be transmitted through the Gaussian channel, thus the receiver gets

$$y = \sqrt{\rho n} x(i) + w \quad (8)$$

where $w \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$ denotes a white Gaussian noise vector and ρ the SNR at the receiver (since $x(i)$ has unit norm). Algorithm 1 presents the decoding procedure in pseudo code. The algorithm performs maximum likelihood (ML) decoding

Algorithm 1 DECODE(y)

```

1: MLcandidates  $\leftarrow \emptyset$ 
2: for  $j = 0; j < 2^{k-1}; j++$  do
3:    $y(j) \leftarrow \text{CREATEVARIANT}(j)$ 
4:    $z(j) \leftarrow \text{SLEPIANDETECT}(y(j))$ 
5:   MLcandidates  $\leftarrow$ 
       MLcandidates  $\cup \{\text{NEWCANDIDATE}(z(j))\}$ 
6: end for
7:  $\hat{i} \leftarrow \text{MLDecode}\{\text{MLcandidates}\}$ 
8: return Message no.  $\hat{i}$ 

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with substantially reduced number of candidate codewords. The careful selection of candidates is the main achievement of the algorithm. It utilizes and refines the detection method SLEPIANDETECT [1] described already in the introduction by creating appropriate variants $y(i)$ of the received sequence y .

Let us go into some detail of Algorithm 1 now and ignore the function CREATEVARIANT for the moment. Then the loop in line 2 becomes trivial, $i = 0$, $y(0) = y$. Recall that the function SLEPIANDETECT would be equivalent to ML decoding if we had taken the full codebook $S_n \hat{x}$ for

^{d)}Note that different from the presentation in [10] all loops and sum boundaries have to be decreased by one for an implementation in C, except for the upper sum boundary l in function Mapping. Moreover, j has to be chosen as the largest l with the property $s_l \geq 0$

transmission. Since our code is a subset, SLEPIANDETECT might fail, but the basic idea of Algorithm 1 is that due to the Gray like ordering of the multiset permutations the obtained candidate is some kind of neighbor (with respect to Euclidean distance) of the transmitted sequence (compare the discussion in section II). This fact is implicitly contained in the definition of the function NEWCANDIDATE, see Algorithm 2: It takes the

Algorithm 2 NEWCANDIDATE(z)

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1:  $lr \leftarrow \text{LEXRANK}(z)$ 
2:  $gr \leftarrow \text{LEX2GRAY}(lr)$ 
3: return  $\lfloor \text{MPGRAYENCODE}^{-1}(gr) \rfloor$ 

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output of SLEPIANDETECT, calculates its lexicographic rank, transforms it with the help of LEX2GRAY (7) and estimates from that number the codeword number by taking the inverse of (6), where its domain has been enlarged to the reals (which is denoted by $\hat{\cdot}$ here in Algorithm 2^{e)}).

Unfortunately the neighborhood assumption is not correct in general since transmission errors may occur everywhere in the sequence and this is where the function CREATEVARIANT in step 3 of Algorithm 1 enters the stage. Nevertheless Algorithm 2 provides codeword candidate numbers from scratch without going through the list of all codewords in \mathcal{C} , thus its computational complexity is very low. The final detection step 7 denotes ML detection with respect to the codewords listed in the set MLcandidates. Since the cardinality of this set is at most 2^{k-1} the final ML detection has low complexity also.

At last let us consider the function CREATEVARIANTS. The idea is, that at least for not too low SNR values, errors occur only in a few places. Recall that since each ordered m_i received values will be equated to μ_i in SLEPIANDETECT, thus an error occurs only, if the smallest or largest of them is perturbed so badly, that SLEPIANDETECT assigns a wrong value μ_j , $i \neq j$ to it. So this defect interchanges the components m_i and $m_i + 1$ in the sorted sequence. The function CREATEVARIANT loops over all $k-1$ such places and interchanges the corresponding components, compare Algorithm 3. This strategy obviously approaches ML performance when the SNR grows.

Algorithm 3 CREATEVARIANT(j)

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1: create binary representation  $j = \sum_{l=1}^k b_l 2^{l-1}$ 
2:  $\forall l | b_l = 1$ : interchange component no.  $m_l$  and  $m_l + 1$  in the sorted version of  $y$ 

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IV. SIMULATIONS AND DISCUSSION

The following codes have been constructed:

- 1) A code with sequence length $n = 25$ with $N = 323$ codewords (supporting a target rate of $1/3$) out of $M = 600$

^{e)}In the spirit of [11] the rounding in the last step in Algorithm 2 corresponds to the decision regions provided by an appropriate lookup table. However, the use of the Gray code ordering provides fast access to the location of z in the lookup table

multiset permutations corresponding to $m = (1, 23, 1)$ and $\mu = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

- 2) This code has sequence length $n = 50$, $N = 1024$ (supporting the target rate $1/5$) out of $M = 2450$ multiset permutations corresponding to $m = (1, 48, 1)$ and $\mu = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.
- 3) The last code has sequence length $n = 100$, $N = 1024$ (supporting the target rate $1/10$) out of $M = 9900$ multiset permutations corresponding to $m = (1, 98, 1)$ and $\mu = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

Their performance is shown in Fig. 1 for the low SNR regime.

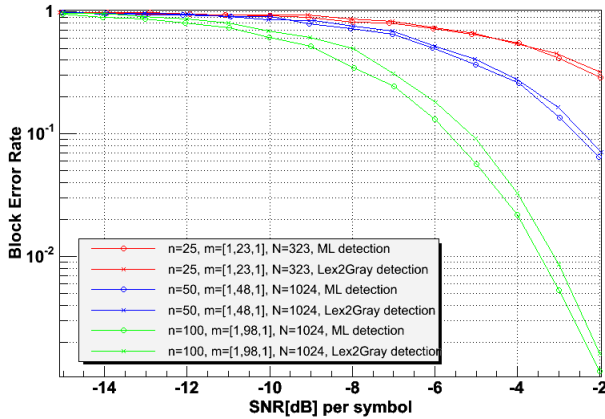


Fig. 1. Performance of codes 1-3 with ML detection and detection corresponding to Algorithm 1

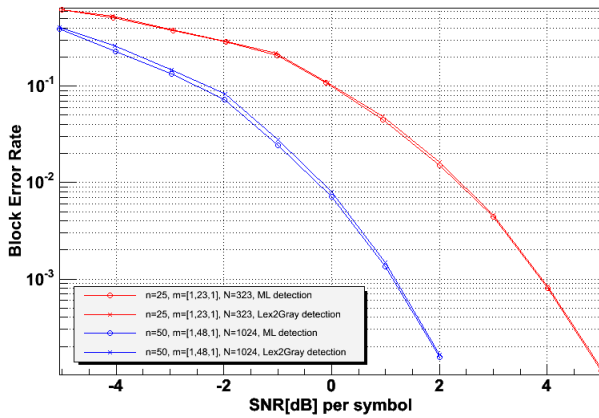


Fig. 2. Performance of codes 1 and 2 in the mid SNR regime with ML detection and detection corresponding to Algorithm 1

The mid SNR regime for the codes 1 and 2 is shown in Fig. 2. The last simulations approve the expectation that the proposed decoding algorithm approaches ML performance when the SNR grows. In fact the performance curves match quite well.

Observe that ML decoding amounts for code 3 1024 comparisons of 100 dimensional vectors, while Algorithm 1 singles out only 4 candidates for ML decoding which are obtained by simple element operations. The complexity of Algorithm 2 is

approximately $nk/2 = 150$, thus the complexity of the whole decoding procedure (Algorithm 1) scales with $n \log(n)$ (due to the sorting operations involved) and is independent of N in contrast to ML decoding which scales with Nn , whereas usually $N \gg n$ holds. As an illustration of computational complexity, table I lists the CPU time for the whole simulation process of codes 1-3 in the SNR range from -15dB to -2dB. Each simulation has been performed on the same machine applying Algorithm 1 and ML decoding respectively. The table reveals 97.5 up to 99 percent saving.

	code 1	code 2	code 3
Algorithm 1	.38	2.22	128.54
ML decoding	14.73	280.46	20346.7
ratio	.0258	.0079	.0063

TABLE I
COMPARISON OF CPU SIMULATION TIMES WITH AND WITHOUT NEAR ML DECODING

In summary, the encoding method described in this paper have been shown to support arbitrary rates with minimum distance lower bounded by the optimal IVP in the sense of [8] with respect to the underlying permutation modulation. Complementary the presented decoding algorithm achieves near ML performance with substantial savings of computation load at the receiver and only small additional memory requirements. By the way a practical implementation for determining the Gray code rank of multiset permutations has been achieved with the help of a simple translation table, which provides fast access. Note also, that encoding and decoding both are simple in structure.

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